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General T-Fractions Corresponding to Functions Satisfying Certain Boundedness Conditions

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1. DEFINITION AND INTRODUCTORY REMARKS

We shall use the symbol K to denote a continued fraction (terminating or non-terminating) in a similar manner as the familiar symbols $\sum$ and $\prod$ are used to denote a sum and a product:

$$\mathop{\mathsf{K}}_{n=1}^{N} \frac{a_n}{b_n} = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_N}{b_N} = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_N}{b_N}}},$$

$$\mathop{\mathsf{K}}_{n=1}^{\infty} \frac{a_n}{b_n} = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots}}.$$

The Nth *approximant* of the nonterminating continued fraction

$$\mathop{\mathsf{K}}_{n=1}^{\infty} \frac{a_n}{b_n}$$

is

$$\mathop{\mathsf{K}}_{n=1}^{N} \frac{a_n}{b_n}.$$

Convergence of a nonterminating continued fraction means convergence of the sequence of approximants.

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A *general T-fraction* is a continued fraction of the form

$$P(z) + Q\left(\frac{1}{z}\right) + c_0 + \mathop{\mathrm{K}}_{n=1}^{N} \frac{F_n z}{1 + G_n z}, \tag{1.1}$$

where P and Q are polynomials $\equiv 0$ or without constant term and where c_0, F_n, G_n are complex numbers, $F_n \neq 0$ for $n < N + 1$. If $N = \infty$, the general T-fraction is called *infinite* or *nonterminating*, if $N < \infty$, it is *finite* or *terminating*. The case $N = 0$, where the K-expression is empty and hence 0, is also accepted.

In the nonterminating case and with $P(z) + Q(1/z) + c_0 = e_0 + d_0 z$, $e_0 \neq 0$ the general T-fraction has the equivalent form

$$e_0 + d_0 z + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{z}{e_n + d_n z}, \qquad e_n \neq 0 \quad \text{for} \quad n \geqslant 0. \tag{1.1'}$$

Continued fractions of this form are referred to by Perron [8, p. 173] as "Thronsche Kettenbrüche," since they are generalizations of the continued fractions

$$1 + d_0 z + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{z}{1 + d_n z}, \tag{1.2}$$

introduced by Thron [9]. Later the continued fractions (1.2) are referred to as T-fractions. In addition to Perron's studies the continued fractions (1.1′) or (1.1) have been studied by McCabe and Murphy (see, e.g., [7]) and independently by Jones and Thron [3].

Some of the most essential properties of general T-fractions have to do with their (possible) correspondence to pairs $(L(z), L^*(z))$ of formal Laurent series and with their relation to two-point Pade tables [3, 5]. The approximants of a general T-fraction (1.1) with $Q \equiv 0$ and $G_n \neq 0$, $n \geqslant 1$ form a diagonal in the two-point Pade table of $(L(z), L^*(z))$. General T-fractions have also proved useful in solving algebraic equations numerically [4] and in solving a certain moment problem [6].

The general T-fraction (1.1) is said to *correspond to* the ordered pair $(L(z), L^*(z))$ of formal Laurent series

$$\begin{aligned} L(z) &= Q\left(\frac{1}{z}\right) + c_0 + \sum_{k=1}^{\infty} c_k z^k, \\ L^*(z) &= P(z) + \sum_{k=-\infty}^{0} c_k^* z^k \end{aligned} \tag{1.3}$$

if and only if P, Q, c_0 are the same in (1.1) and (1.3), and for any natural number n the nth approximant of (1.1) has a Laurent expansion at 0 with the start

$$Q\left(\frac{1}{z}\right) + c_0 + \sum_{k=1}^{n} c_k z^k \qquad \text{(correspondence at 0)},$$

and a Laurent expansion at ∞ with the start

$$P(z) + \sum_{k=-(n-1)}^{0} c_k^* z^k \qquad \text{(correspondence at } \infty\text{)},$$

where c_k and c_k^* have the same meaning as in (1.3) [3]. (In the termininating case all nth approximants for $n \geqslant N$ are equal to the Nth approximant.) In paper [3] a set of necessary and sufficient conditions on the coefficients of $L(z)$ and $L^*(z)$ for existence of a corresponding general, nonterminating T-fraction with all $G_n \neq 0$ is given. It is easy to prove that $G_n \neq 0$ for all $n < N + 1$ is necessary (and sufficient) for a general T-fraction to correspond to some pair (1.3) of formal Laurent series [13], and hence the condition $G_n \neq 0$ does not represent any restriction. However, the theorem in [3] does not cover the terminating case.

It is easy to prove that the pair (1.3) corresponds to the general T-fraction (1.1) if and only if

$$\left(1 + \sum_{k=1}^{\infty} c_k z^k - P(z), 1 - c_0 + c_0^* + \sum_{k=-\infty}^{-1} c_k^* z^k - Q\left(\frac{1}{z}\right)\right)$$

corresponds to

$$1 + \mathop{\mathrm{K}}_{n=1}^{N} \frac{F_n z}{1 + G_n z}.$$

Hence, without loss of generality we may restrict ourselves to formal Laurent series of the types

$$L(z) = 1 + \sum_{k=1}^{\infty} c_k z^k, \qquad L^*(z) = \sum_{k=-\infty}^{0} c_k^* z^k, \tag{1.4}$$

and general T-fractions of the form

$$1 + \mathop{\mathrm{K}}_{n=1}^{N} \frac{F_k z}{1 + G_k z} \qquad [13]. \tag{1.5}$$

Let $(L(z), L^*(z))$ be a pair of formal power series (1.4), and let for a fixed n, $1 \leqslant n < N+1$,

$$1 + \mathop{\mathrm{K}}_{k=1}^{n} \frac{F_k z}{1 + G_k z} \tag{1.6}$$

be a terminating general T-fraction with the property that for any integer k, $1 \leqslant k \leqslant n$, the kth approximant of (1.6) has a Laurent expansion at 0 with the start

$$1 + \sum_{\nu=1}^{k} c_\nu z^\nu$$

and a Laurent expansion at ∞ with the start

$$\sum_{\nu=-(k-1)}^{0} c_\nu^* z^\nu,$$

c_ν and c_ν^* being the same as in (1.4). Then (1.6) shall be called a *proper start* (a proper *n-start* if we want to emphasize the length) of a possible general T-fraction corresponding to $(L(z), L^*(z))$, or briefly a proper start for $(L(z), L^*(z))$. It is obvious, that except for the case $L(z) = L^*(z) = 1$, when the corresponding general T-fraction is 1, a necessary condition for existence of a corresponding general T-fraction is the existence of proper starts of any length or a proper start which is the (terminating) general T-fraction corresponding to $(L(z), L^*(z))$. It is rather easy to prove that this is also sufficient (A crucial point is the uniqueness of the parameters F_k, G_k) [13].

It is a straightforward verification to prove that for any $(L(z), L^*(z)) \neq (1, 1)$ a proper 1-start exists if and only if $c_1 \neq 0$, $c_0^* \neq 1$, in which case

$$1 + \frac{F_1 z}{1 + G_1 z} \quad \text{with } F_1 = c_1, \; G_1 = \frac{c_1}{c_0^* - 1} \tag{1.7}$$

is the unique proper 1-start. If a proper 1-start exists, the formal identities

$$L(z) = 1 + \frac{F_1 z}{G_1 z + l(z)}, \qquad L^*(z) = 1 + \frac{F_1 z}{G_1 z + l^*(z)} \tag{1.8}$$

define a new pair $(l(z), l^*(z))$ of formal Laurent series of the form (1.4). This pair is in [13] called the *descendant* of $(L(z), L^*(z))$. In the particular case when $(l(z), l^*(z)) = (1, 1)$ the terminating general T-fraction (1.7) is the corresponding general T-fraction of $(L(z), L^*(z))$. In all other cases,

$(l(z), l^*(z))$ may or may not have a proper 1-start. It is easy to prove that the following holds for any $n \geqslant 2$:

$$1 + \mathop{\mathrm{K}}_{k=1}^{n} \frac{F_k z}{1 + G_k z}$$

is a proper n-start for $(L(z), L^*(z))$ if and only if

$$1 + \mathop{\mathrm{K}}_{k=2}^{n} \frac{F_k z}{1 + G_k z}$$

is a proper $(n - 1)$-start for $(l(z), l^*(z))$ [13].

2. A Boundedness $\Rightarrow$ Convergence Theorem

In paper [10] it is, for ordinary T-fractions, proved that if f is holomorphic in a sufficiently large disk $|z| < R$, $f(0) = 1$, and $|f(z) - 1|$ is sufficiently small in the disk, then the corresponding T-fraction converges to $f(z)$ locally uniformly on the unit disk $|z| < 1$. In paper [11] a related theorem for correspondence at ∞ is proved. In both cases the T-fraction turns out to be limit periodic with $d_n \to -1$ as $n \to \infty$.

The purpose of the present paper is to prove that a similar connection between boundedness and convergence exists for general T-fractions. Since in case of correspondence the general T-fraction is governed by *two* formal series, it is likely that we must put conditions on both. It turns out that if $L(z)$ and $L^*(z)$ are Laurent series of *functions* satisfying sufficiently strong boundedness conditions, then the corresponding general T-fraction will exist and will converge to the "right" functions in neighborhoods of 0 and ∞.

Notational Remarks. In the following L and L^* shall denote the functions having $L(z)$ and $L^*(z)$ as their Laurent expansions (at 0 and at ∞). Furthermore, for any fixed c_1, we shall let $\mathscr{L}_{c_1}$ denote the set of formal Laurent series

$$1 + c_1 z + c_2 z^2 + \cdots$$

with that particular value of c_1.

THEOREM. *For any fixed $c_1 \neq 0$ there exist two ordered pairs (α, R), (β, ρ) of positive numbers,*

$$\rho < \frac{1}{|c_1|} < R,$$

such that the following holds for all $(L(z), L^*(z))$ with $L(z) \in \mathscr{L}_{c_1}$:

If

$$L \text{ is a holomorphic function in the disk } |z| < R \text{ and there satisfies the condition} \tag{2.1}$$

$$|L(z) - 1 - c_1 z| \leqslant \alpha$$

and

$$L^* \text{ is a holomorphic function in the domain } |z| > \rho \text{ and there satisfies the condition} \tag{2.2}$$

$$|L^*(z)| \leqslant \beta,$$

then $(L(z), L^*(z))$ *has a corresponding nonterminating general T-fraction*

$$1 + \operatorname*{K}_{n=1}^{\infty} \frac{F_n z}{1 + G_n z}$$

with the property that

$$\lim_{n\to\infty} F_n = - \lim_{n\to\infty} G_n = F$$

exists and is $\neq 0$. *The general T-fraction converges to* $L(z)$ *locally uniformly in* $|z| < 1/|F|$ *and to* $L^*(z)$ *locally uniformly in* $|z| > 1/|F|$.

Before proving the theorem we shall make some remarks:

Remark 1. For any fixed $c_1 \neq 0$ the pair $(L(z), L^*(z)) = (1 + c_1 z, 0)$ obviously satisfies the conditions (2.1) and (2.2), regardless of the values of α, R, β, ρ. It is easy to prove, that it has the corresponding general T-fraction

$$1 + \frac{c_1 z}{1 - c_1 z} + \frac{c_1 z}{1 - c_1 z} + \frac{c_1 z}{1 - c_1 z} + \cdots$$

and that this converges to $1 + c_1 z$ locally uniformly in $|z| < 1/|c_1|$ and to 0 locally uniformly in $|z| > 1/|c_1|$. The theorem expresses the fact that if $(L(z), L^*(z))$ is close to $(1 + c_1 z, 0)$ in a certain sense, then it acts similarly as far as the corresponding general T-fraction is concerned.

Remark 2. If α, R, β, ρ "work" in the sense of the theorem, and α', R', β' ρ' are positive numbers with $\alpha' \leqslant \alpha$, $R' \geqslant R$, $\beta' \leqslant \beta$, $\rho' \leqslant \rho$ they will also "work" in the same meaning.

Remark 3. To *any* pair of series (1.4), which are Laurent expansions of functions, we can construct a pair $(\tilde{L}, \tilde{L}^*)$ of functions, meeting the requirements (2.1) and (2.2). In fact

$$\tilde{L}(z) = L(kz) + (1 - k)\, c_1 z,$$

$$\tilde{L}^*(z) = L^*\left(\frac{z}{k}\right)$$

will do for all sufficiently small $|k|$. Observe that this transformation does not change c_1.

Remark 4. Let a and b be distinct complex numbers $\neq 0$, and let

$$L(z) = \frac{1 - abz^2}{1 - az}, \qquad L^*(z) = 0.$$

Then obviously L^* satisfies the condition (2.2), regardless of the values of ρ and β. We also see, that $c_1 = a$, and that

$$L(z) - 1 - az = (a - b)\frac{az^2}{1 - az}.$$

On any fixed disk $|z| < R < 1/|a|$ this can be made arbitrarily small by taking b to be sufficiently close to a. This, however, is not enough to satisfy (2.1), since R cannot be taken to be $> 1/|c_1| = 1/|a|$. It is easy to see that the corresponding general T-fraction exists and is of the form

$$1 + \frac{az}{1 - az} + \frac{bz}{1 - bz} + \frac{az}{1 - az} + \frac{bz}{1 - bz} + \cdots$$

Since $a \neq b$, this is obviously not limit periodic.

Proof of the Theorem. Without loss of generality we may assume that $|c_0^*| < 1$ and hence $c_0^* \neq 1$. Since we already have required $c_1 \neq 0$, we know that $(L(z), L^*(z))$ has a proper start

$$1 + \frac{F_1 z}{1 + G_1 z}, \qquad F_1 = c_1, \qquad G_1 = \frac{c_1}{c_0^* - 1},$$

and a descendant $(l(z), l^*(z))$:

$$\begin{aligned} l(z) &= 1 + d_1 z + d_2 z^2 + \cdots, \\ l^*(z) &= d_0^* + d_{-1}^* z^{-1} + d_{-2}^* z^{-2} + \cdots. \end{aligned} \tag{2.3}$$

In the proof we shall need the following formulas, which are proved by straightforward computations:

$$d_1 = \frac{c_1}{1 - c_0^*} - \frac{c_2}{c_1}, \tag{2.4}$$

$$l(z) - 1 - d_1 z = \frac{c_2}{c_1} z - \frac{\dfrac{L(z) - 1 - c_1 z}{c_1 z}}{\dfrac{L(z) - 1 - c_1 z}{c_1 z} + 1}, \tag{2.5}$$

$$d_0^* = -\frac{c_1 c_{-1}^*}{(1 - c_0^*)^2}, \tag{2.6}$$

$$l^*(z) = \frac{(L^*(z) - c_0^*)\, c_1 z}{(L^*(z) - 1)(1 - c_0^*)}. \tag{2.7}$$

Let m and M be arbitrary positive numbers such that

$$m < | c_1 | < M, \tag{2.8}$$

and let θ be an arbitrary positive number <1. After having chosen m, M, and θ, let α, R, β, ρ be positive numbers such that $\beta < 1$ and

$$mR \geqslant \alpha + \frac{2}{\theta}, \tag{2.9a}$$

$$\frac{\alpha}{mR^2} + \frac{\beta M}{1 - \beta} \leqslant (1 - \theta) \cdot \min\{M - | c_1 |, | c_1 | - m\}, \tag{2.9b}$$

$$\frac{2M\rho}{(1 - \beta)^2} \leqslant \theta. \tag{2.9c}$$

The existence of such numbers is trivial. We also see, that once such a quadruple is determined, any quadruple $(\alpha', R', \beta', \rho')$ of positive numbers with

$$\alpha' \leqslant \alpha, \qquad R' \geqslant R, \qquad \beta' \leqslant \beta, \qquad \rho' \leqslant \rho$$

will satisfy (2.9). Furthermore, it follows from (2.9a) that $R > 1/| c_1 |$ and from (2.9c) that $\rho < 1/| c_1 |$. (We even see that $R > 2/| c_1 |$ and $\rho < 1/(2 | c_1 |)$ for any R, ρ satisfying the set (2.9) of inequalities.)

Assume now that L and L^* satisfy the conditions (2.1) and (2.2) with the values just chosen for α, R, β, ρ. From (2.1) it follows, by using Schwarz' Lemma twice:

$$\left| \frac{L(z) - 1 - c_1 z}{z} \right| \leqslant \frac{\alpha}{R} \quad \text{in} \quad | z | < R, \tag{2.10}$$

$$\left| \frac{L(z) - 1 - c_1 z}{z^2} \right| \leqslant \frac{\alpha}{R^2} \quad \text{in} \quad | z | < R. \tag{2.11}$$

This also holds at the origin (by removing the singularity in the usual way), and hence we have

$$| c_2 | \leqslant \frac{\alpha}{R^2}. \tag{2.11'}$$

From (2.5), (2.10), (2.11′), and (2.9a) it follows that l is holomorphic in $|z| < R$ and there satisfies the inequality

$$|l(z) - 1 - d_1 z| \leqslant \frac{\alpha}{mR} + \frac{\alpha}{mR - \alpha} \leqslant \theta \cdot \alpha. \tag{2.12}$$

From (2.4), (2.2) with $z \to \infty$, (2.11′), and (2.9b) it follows that

$$\begin{aligned} |d_1 - c_1| &= \left| \frac{c_1 c_0^*}{1 - c_0^*} - \frac{c_2}{c_1} \right| \leqslant \frac{M\beta}{1-\beta} + \frac{\alpha}{mR^2} \\ &\leqslant (1-\theta) \cdot \min\{M - |c_1|, |c_1| - m\}. \end{aligned} \tag{2.13}$$

From (2.2) we have

$$|L^*(z) - c_0^*| \leqslant 2\beta \qquad \text{in} \quad |z| > \rho$$

and

$$L^*(z) - c_0^* \to 0.$$

An "exterior domain version" of Schwarz's Lemma (obtained by using Schwarz's Lemma on $L^*(1/\zeta) - c_0$ in the disk $|\zeta| < 1/\rho$) thus gives

$$|(L^*(z) - c_0^*)z| \leqslant 2\beta\rho \qquad \text{in} \quad |z| > \rho,$$

and hence we have from (2.7) and (2.9c) that l^* is holomorphic in $|z| > \rho$ and

$$|l^*(z)| \leqslant \frac{2\beta M\rho}{(1-\beta)^2} \leqslant \theta \cdot \beta. \tag{2.14}$$

Conditions (2.12) and (2.14) show that the descendant $(l(z), l^*(z))$ satisfies the conditions (2.1) and (2.2) with (α, β) replaced by $(\theta\alpha, \theta\beta)$. Furthermore, from (2.13) it follows that

$$m < |d_1| < M.$$

The inequalities (2.9) obviously hold when (α, β) is replaced by $(\theta\alpha, \theta\beta)$, (2.9b) even with

$$\theta(1-\theta) \cdot \min\{M - |c_1|, |c_1| - m\} \leqslant (1-\theta) \cdot \min\{M - |d_1|, |d_1| - m\}$$

on the right-hand side. Hence we may repeat the argument and obtain a descendant of $(l(z), l^*(z))$, satisfying (2.1) and (2.2) with (α, β) replaced by $(\theta^2\alpha, \theta^2\beta)$. This process may be continued as long as we know that the coefficient of z in the first of the two series always is in the (m, M)-annulus (2.8). Denoting the possible kth descendant by $(L^{(k)}(z), L^{*(k)}(z))$,

$$L^{(k)}(z) = 1 + \sum_{n=1}^{\infty} c_n^{(k)} z^n, \qquad L^{*(k)}(z) = \sum_{n=-\infty}^{0} c_n^{*(k)} z^n,$$

we have

$$\begin{aligned} | c_1^{(k)} - c_1 | &\leqslant | c_1^{(1)} - c_1 | + | c_1^{(2)} - c_1^{(1)} | + \cdots + | c_1^{(k)} - c_1^{(k-1)} | \\ &\leqslant (1-\theta)(1 + \theta + \cdots + \theta^{k-1}) \cdot \min\{M - | c_1 |, | c_1 | - m\} \\ &< \min\{M - | c_1 |, | c_1 | - m\}. \end{aligned}$$

Hence $m < | c_1^{(k)} | < M$ for all k, and all descendants exist, and hence the corresponding general T-fraction. Since the series $\sum | c_1^{(k)} - c_1^{(k-1)} |$ converges, the sequence $\{c_1^{(k)}\}$ must converge, and since $F_k = c_1^{(k)}$, we have

$$\lim_{k\to\infty} F_k = F, \qquad \text{where} \quad m \leqslant | F | \leqslant M.$$

Since furthermore $| c_0^{*(k)} | \leqslant \theta^k \cdot \beta$ and

$$G_k = \frac{c_1^{(k)}}{c_0^{*(k)} - 1},$$

we have

$$\lim_{k\to\infty} G_k = -F.$$

The "convergence part" of the statement in the theorem is a simple consequence of a theorem in Perron [8, p. 93] on limit periodic continued fractions. The argument is very similar to the one given in [10], and shall be omitted here.

3. Final Remarks

Let $c_1 \neq 0$ be given, and let R and ρ be any two positive numbers such that

$$R > \frac{2}{| c_1 |}, \qquad \rho < \frac{1}{2 | c_1 |}.$$

Then there exist numbers $\alpha > 0$, $\beta > 0$, such that if $L(z) \in \mathscr{L}_{c_1}$ and (2.1) and (2.2) are satisfied, then the corresponding general T-fraction exists, is limit periodic with $F_n \to F \neq 0$, $G_n \to -F$, and converges to $L(z)$ locally uniformly on $| z | < 1/| F |$ and to $L^*(z)$ locally uniformly on $| z | > 1/| F |$.

In order to see this we may proceed as follows: Pick m, M such that

$$\frac{2}{R} < m < |c_1| < M < \frac{1}{2\rho},$$

and pick $\theta \in (0, 1)$ such that

$$\frac{2}{R\theta} < m \quad \text{and} \quad \frac{M}{\theta} < \frac{1}{2\rho}.$$

Then, for any $\alpha > 0$ with $\alpha/R \leqslant m - 2/R\theta$ (2.9a) holds, and for any $\beta > 0$ with $\beta < 1$ and $(1-\beta)^2 \geqslant 2M\rho/\theta$ (2.9c) holds. Observe that if (2.9a) and (2.9c) hold for a certain pair (α, β), they hold for all (α', β'), where $0 < \alpha' < \alpha$, $0 < \beta' < \beta$. Since the left-hand side expression of (2.9b) tends to 0 when α and β *both* tend to 0, (2.9b) holds for all sufficiently small α, β. Hence, if $R > 2/|c_1|$ and $\rho < 1/(2|c_1|)$, it is possible to find numbers $m, M, \theta, \alpha, \beta$, such that (2.8) and (2.9) hold.

The proof of the theorem does not offer any values of R and ρ better than $2/|c_1|$ and $1/(2|c_1|)$. Based upon the Hovstad result [2] on ordinary T-fractions a nearby conjecture would be that any pair (R, ρ) with

$$\rho < \frac{1}{|c_1|} < R$$

works.

For a limit-periodic continued fraction the use of converging factors [14] is usually worth a try, since it may increase the domain of convergence and the speed of the convergence. This has been done successfully for ordinary T-fractions in [12], and is discussed more generally by Gill in several papers, e.g., [1]. In the present case the method would mean to replace $S_n(0)$ by $S_n(Fz)$ at 0 (or rather $S_n(F_n z)$, since F usually is not known), and to replace $S_n(0)$ by $S_n(-1)$ at ∞, where

$$S_n(w) = 1 + \frac{F_1 z}{1 + G_1 z} + \cdots + \frac{F_n z}{1 + G_n z + w}.$$

Numerical examples seem to indicate that in some cases very much can be gained by such a modification.

EXAMPLE. It can be proved, that the continued fraction

$$1 + \operatorname*{K}_{k=1}^{\infty} \frac{z}{1 - \left(1 + \frac{1}{5^{k+1}}\right) z} \tag{3.1}$$

corresponds at 0 to a series, representing a function f, meromorphic in the disk $|z| < 5$. It can furthermore be proved, that the continued fraction (3.1) converges to $f(z)$ in $|z| < 1$ and in no larger disk, but that the sequence $\{S_n(z)\}$ converges to $f(z)$ in $|z| < 5$ (except at possible poles).

As for the speed of convergence, the values

$$f(-0.9) = 0.498, \qquad f(0.9) = 1.568,$$

rounded in the 3rd decimal place need $n = 46$ and $n = 80$ in the usual continued fraction algorithm (i.e., $S_n(0)$), whereas they are obtained already for $n = 3$ with the modification (i.e., $S_n(z)$).

The use of converging factors will be discussed in more detail in a subsequent paper.

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